Characterization and Estimation of Double Weighted Rayleigh Distribution

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Abstract

This paper presents a new weighted distribution which is known as the Double Weighted Rayleigh Distribution (DWRD). This distribution is constructed and studied. The statistical properties of this distribution are discussed and obtained, including the mean, variance, coefficient of variation, harmonic mean, moments, mode, reliability function, hazard function and the reverse hazard function. Also the parameters of this distribution are estimated by the method of moment and the maximum likelihood estimation method.

Keywords: Weighted distribution, Double Weighted distribution, Rayleigh distribution

1. Introduction

Weighted distribution provides an approach to dealing with model specification and data interpretation problems. Fisher [3] and Rao [10] introduced and unified the concept of weighted distribution. Fisher [3] studied on how methods of ascertainment can influence the form of distribution of recorded observations and then Rao [10] introduced and formulated it in general terms in connection with modeling statistical data where the usual practice of using standard distributions for the purpose was not found to be appropriate. In Rao's paper, he identified various situations that can be modeled by weighted distributions. These situations refer to instances where the recorded observations cannot be considered as a random sample from the original distributions. This may occur due to non-observability of some events or damage caused to the original observation resulting in a reduced value, or adoption of a sampling procedure which gives unequal chances to the units in the original.

Weighted distributions occur frequently in research related to reliability, bio-medicine, ecology and branching process can be seen in Gupta and Kirmani [5], Patil and Rao [13], Gupta and Keating [4]. Das and Roy [2] discussed the length-biased Weighted Generalized Rayleigh distribution with its properties, also they develop the length-biased Weighted Weibull distribution. Patil and Ord [12] introduced the concept of size biased sampling and weighted distributions by identifying some of the situations where the underlying models retain their form. Kareema and Hantoosh [7] studied the Double Weighted Exponential distribution and some of its structural properties. Shi et al [14] studied the theoretical properties of weighted Generalized Rayleigh and related distributions. Rashwan [11] presented the generalized gamma length biased distribution with its properties. For more important results of weighted distributions see Oluyede and Terbeche [9], Oluyede and George [8], Jing [6].

Suppose X is a non-negative random variable with probability density function (pdf) $f(x)$, then the pdf of the weighted random variable X_w is given by

$$f_w(x)=\frac{w(x)f(x)}{E(w(x))}\ ,\ x>0$$

where $w(x)$ be a non-negative weight function.

Depending upon the choice of the weight function $w(x)$, we have different weighted models. Clearly when $w(x)=x$, the resulting distribution is called length-biased whose pdf is given by:

$$f_l(x)=\frac{xf(x)}{E(x)}\ ,\ x>0$$

In this paper we present the Double Weighted distribution (DWD), taking one type of weight function $w(x)=x$, and using the Rayleigh distribution as original distribution, we derive the pdf and some useful properties of Double Weighted Rayleigh distribution.

2. Double Weighted Distribution

Definition (2.1): The Double Weighted distribution (DWD) is given by:

$$f_w(x;c)=\frac{w(x)\ f(x)F(cx)}{W_D}\quad ,\quad x\geq 0,\ c>0$$

where $W_D=\int_0^\infty w(x)f(x)F(cx)dx$

and the first weight is $w(x)$ and the second is (cx). $F(cx)$ depend on the original distribution $f(x)$.

3. Double Weighted Rayleigh Distribution

Consider the first weight function $w(x)=x$ and the pdf of Rayleigh distribution is given by:

$$f(x,\theta)=\frac{x}{\theta^2}\exp\left(-\frac{x^2}{2\theta^2}\right),\ \ x\geq 0,\theta>0 \qquad (1)$$

and $F(cx;\theta)=1-\exp\left(-\frac{c^2x^2}{2\theta^2}\right),\ c>0$

and $W_D=\int_0^\infty w(x)f(x)F(cx)dx\ =\int_0^\infty \frac{x^2}{\theta^2}\exp\left(-\frac{x^2}{\theta^2}\right)\left\{1-\exp\left(-\frac{(cx)^2}{2\theta^2}\right)\right\}dx$

$$\Rightarrow W_D=\int_0^\infty w(x)f(x)F(cx)dx\ =\sqrt{\frac{\pi}{2}}\,\theta\,\frac{\left((1+c^2)^{\frac{3}{2}}-1\right)}{(1+c^2)^{\frac{3}{2}}}$$

Then the pdf of Double Weighted Rayleigh Distribution (DWRD) is:

$$g_w(x;\theta,c)=\sqrt{\frac{2}{\pi}}\frac{(1+c^2)^{\frac{3}{2}}}{\left((1+c^2)^{\frac{3}{2}}-1\right)}\frac{x^2}{\theta^3}\exp\left(-\frac{x^2}{2\theta^2}\right)\left\{1-\exp\left(-\frac{(cx)^2}{2\theta^2}\right)\right\}\ ,\ x\geq 0,c>0 \qquad (2)$$

and its Cumulative distribution function (cdf) is given by

$$G_w(x;\theta,c)=\sqrt{\frac{2}{\pi}}\frac{(1+c^2)^{\frac{3}{2}}}{\left((1+c^2)^{\frac{3}{2}}-1\right)}\frac{1}{\theta}\left(\int_0^x\frac{t^2}{\theta^2}\exp\left(-\frac{t^2}{2\theta^2}\right)dt-\int_0^x\frac{t^2}{\theta^2}\exp\left(-\frac{t^2}{2\theta^2}(1+c^2)\right)dt\right)$$

$$=\frac{2(1+c^2)^{\frac{3}{2}}}{\left((1+c^2)^{\frac{3}{2}}-1\right)\sqrt{\pi}}\left(\Gamma\left(\frac{3}{2},\frac{x^2}{2\theta^2}\right)-\Gamma\left(\frac{3}{2},\frac{x^2}{2\theta^2}(1+c^2)\right)\right)$$

4. Reliability Analysis

(i) Reliability function *R*(*x*)

The reliability function or survival function *R*(*x*). This function can be derived using the cumulative distribution function and is given by

$$R(x) = 1 - G_w(x)$$

$$= 1 - \frac{2(1+c^2)^{\frac{3}{2}}}{\left((1+c^2)^{\frac{3}{2}} - 1\right)\sqrt{\pi}} \left(\Gamma\left(\frac{3}{2}, \frac{x^2}{2\theta^2}\right) - \Gamma\left(\frac{3}{2}, \frac{x^2}{2\theta^2}(1+c^2)\right)\right)$$

$$= \frac{\left((1+c^2)^{\frac{3}{2}} - 1\right)\sqrt{\pi} - 2(1+c^2)^{\frac{3}{2}} \left(\Gamma\left(\frac{3}{2}, \frac{x^2}{2\theta^2}\right) - \Gamma\left(\frac{3}{2}, \frac{x^2}{2\theta^2}(1+c^2)\right)\right)}{\left((1+c^2)^{\frac{3}{2}} - 1\right)\sqrt{\pi}}$$

(ii) Hazard Function *H*(*x*)

The hazard or instantaneous rate function is denoted by *H*(*x*). The hazard function of *x* can be interpreted as instantaneous rate or the conditional probability density of failure at time *x*, given that the unit has survived until *x*. The hazard function is defined to be

$$H(x) = \frac{g_w(x)}{R(x)}$$

$$= \frac{\sqrt{2}(1+c^2)^{\frac{3}{2}} \frac{x^2}{\theta^3} \exp\left(-\frac{x^2}{2\theta^2}\right)\left\{1 - \exp\left(-\frac{(cx)^2}{2\theta^2}\right)\right\}}{\left((1+c^2)^{\frac{3}{2}} - 1\right)\sqrt{\pi} - 2(1+c^2)^{\frac{3}{2}} \left(\Gamma\left(\frac{3}{2}, \frac{x^2}{2\theta^2}\right) - \Gamma\left(\frac{3}{2}, \frac{x^2}{2\theta^2}(1+c^2)\right)\right)}$$

(iii) Reverse Hazard function $\phi(x)$

The reverse hazard function can be interpreted as an approximate probability of failure in [*x* ، *x* + *dx*], given that the failure had occurred in [0, x]. The reverse hazard function $\phi(x)$ is defined to be

$$\phi(x) = \frac{g_w(x)}{G_w(x)} = \frac{\frac{x^2}{\sqrt{2}\theta^3} \exp\left(-\frac{x^2}{2\theta^2}\right)\left\{1 - \exp\left(-\frac{(cx)^2}{2\theta^2}\right)\right\}}{\left(\Gamma\left(\frac{3}{2}, \frac{x^2}{2\theta^2}\right) - \Gamma\left(\frac{3}{2}, \frac{x^2}{2\theta^2}(1+c^2)\right)\right)}$$

5. The Statistical Properties of DWRD

In this section, we present the statistical properties of DWRD throughout computing the mean, variance, standard deviation, coefficient of variation, harmonic mean, moments and mode as follow:

Moments of DWRD

The *r*th moment of DWRD is given by

$$E(X^r) = \mu'_r = \int_0^\infty x^r g_w(x;\theta,c)\,dx$$

$$=\sqrt{\frac{2}{\pi}}\frac{\left(1+c^2\right)^{\frac{3}{2}}}{\left(\left(1+c^2\right)^{\frac{3}{2}}-1\right)}\int_0^\infty x^r\frac{x^2}{\theta^3}\exp\left(-\frac{x^2}{2\theta^2}\right)\left\{1-\exp\left(-\frac{(cx)^2}{2\theta^2}\right)\right\}dx$$

$$=\frac{2^{\frac{r}{2}+1}\theta^r\left(1+c^2\right)^{\frac{3}{2}}}{\sqrt{\pi}\left(\left(1+c^2\right)^{\frac{3}{2}}-1\right)}\left(\Gamma\left(\frac{r}{2}+\frac{3}{2}\right)-\frac{\Gamma\left(\frac{r}{2}+\frac{3}{2}\right)}{\left(1+c^2\right)^{\frac{r}{2}+\frac{3}{2}}}\right) \qquad (3)$$

Then from equation (3), we can find mean, variance, standard deviation, coefficient of variation as follows:

Mean:

$$\mu=\frac{2^{\frac{3}{2}}\theta\left(\left(1+c^2\right)^2-1\right)}{\sqrt{\pi}\left(\left(1+c^2\right)^{\frac{3}{2}}-1\right)\left(1+c^2\right)^{\frac{1}{2}}} \qquad (4)$$

$$E(x^2)=\mu_2'=\frac{3\theta^2\left(\left(1+c^2\right)^{\frac{5}{2}}-1\right)}{\left(\left(1+c^2\right)^{\frac{3}{2}}-1\right)\left(1+c^2\right)}$$

Variance:

$$\sigma^2=\frac{\theta^2\left\{3\pi\left(\left(1+c^2\right)^{\frac{5}{2}}-1\right)\left(\left(1+c^2\right)^{\frac{3}{2}}-1\right)-8\left(\left(1+c^2\right)^2-1\right)^2\right\}}{\pi\left(1+c^2\right)\left(\left(1+c^2\right)^{\frac{3}{2}}-1\right)^2} \qquad (5)$$

Standard deviation:

$$\sigma=\sqrt{\frac{\theta^2\left\{3\pi\left(\left(1+c^2\right)^{\frac{5}{2}}-1\right)\left(\left(1+c^2\right)^{\frac{3}{2}}-1\right)-8\left(\left(1+c^2\right)^2-1\right)^2\right\}}{\pi\left(1+c^2\right)\left(\left(1+c^2\right)^{\frac{3}{2}}-1\right)^2}} \qquad (6)$$

Coefficient of variation:

$$C.V=\frac{\sigma}{\mu}=\frac{\sqrt{\theta\left\{3\pi\left(\left(1+c^2\right)^{\frac{5}{2}}-1\right)\left(\left(1+c^2\right)^{\frac{3}{2}}-1\right)-8\left(\left(1+c^2\right)^2-1\right)^2\right\}}}{2^{\frac{3}{2}}\left(\left(1+C^2\right)^2-1\right)} \qquad (7)$$

Harmonic Mean:

$$\frac{1}{H}=E\left(\frac{1}{X}\right)=\int_{o}^{\infty}\frac{1}{x}g_w(x;\theta,c)\,dx$$

$$\frac{1}{H}=\sqrt{\frac{2}{\pi}}\frac{\left(1+C^2\right)^{\frac{1}{2}}}{\theta\left(\left(1+C^2\right)^{\frac{3}{2}}-1\right)}$$

$$\therefore H=\sqrt{\frac{\pi}{2}}\frac{\theta\left(\left(1+C^2\right)^{\frac{3}{2}}-1\right)}{\left(1+C^2\right)^{\frac{1}{2}}} \qquad (8)$$

Moment Generating Function:

The moment generating function of DWRD is given by

$$M_X(t)=\int_0^{\infty}e^{tx}\,g_w(x;\theta,c)\,dx$$

$$=\int_0^{\infty}\left(1+tx+\frac{(tx)^2}{2!}+...\right)f(x,\theta,\lambda)\,dx$$

$$=\int_0^{\infty}\sum_{r=0}^{\infty}\frac{t^r}{r!}x^r\ f(x;\theta,c)\,dx$$

$$=\sum_{r=0}^{\infty}\frac{t^r}{r!}E(X^r)$$

$$=\sum_{r=0}^{\infty}\frac{t^r}{r!}\frac{2^{\frac{r}{2}+1}\theta^r\left(1+c^2\right)^{\frac{3}{2}}}{\sqrt{\pi}\left(\left(1+c^2\right)^{\frac{3}{2}}-1\right)}\left(\Gamma\left(\frac{r}{2}+\frac{3}{2}\right)-\frac{\Gamma\left(\frac{r}{2}+\frac{3}{2}\right)}{\left(1+c^2\right)^{\frac{r}{2}+\frac{3}{2}}}\right)$$

Mode of the function

The probability density function of DWRD is

$$g_w(x;\theta,c)=\sqrt{\frac{2}{\pi}}\frac{\left(1+c^2\right)^{\frac{3}{2}}}{\left(\left(1+c^2\right)^{\frac{3}{2}}-1\right)}\frac{x^2}{\theta^3}\exp\left(-\frac{x^2}{2\theta^2}\right)\left\{1-\exp\left(-\frac{(cx)^2}{2\theta^2}\right)\right\},\ x\geq 0, c>0$$

In order to discuss monotonicity of DWRD, we take the logarithm of its pdf as follows:

$$\log g_w(x;\theta,c)=\log k+2\log x-\log\theta^3-\frac{x^2}{2\theta^2}+\log\left\{1-\exp\left(-\frac{(cx)^2}{2\theta^2}\right)\right\}$$

Where k is constant

Differentiating the above equation with respect to x and equating to zero, we obtain

$$\frac{\partial\log g_w(x;\theta,c)}{\partial x}=\frac{2}{x}-\frac{x}{\theta^2}+\frac{c^2x\exp\left(-\frac{(cx)^2}{2\theta^2}\right)}{\theta^2\left\{1-\exp\left(-\frac{(cx)^2}{2\theta^2}\right)\right\}}$$

The mode of the DWRD is obtained by solving the following non-linear equation with respect to *x*.

$$\frac{2}{x}-\frac{x}{\theta^2}+\frac{c^2 x \exp\left(-\frac{(cx)^2}{2\theta^2}\right)}{\theta^2\left\{1-\exp\left(-\frac{(cx)^2}{2\theta^2}\right)\right\}}=0 \qquad (9)$$

Table 1: Mean, Variance (VAR), Standard Deviation (STD), Coefficient of Variation (C.V)

θ	*c*	Mean	VAR	STD	C.V
1	1	1.8513	0.3927	0.6266	0.3384
	2	1.6824	0.4052	0.6365	0.3783
	5.2	1.6045	0.4450	0.6670	0.4157
	7	1.5996	0.4494	0.6703	0.4190
2	1	3.7027	1.5708	1.2533	0.4786
	2	3.3648	1.6208	1.2731	0.5350
	5.2	3.2090	1.78019	1.3342	0.5879
	7	3.1993	1.7977	1.3407	0.5926
6.5	1	12.0340	16.5924	4.0733	0.8629
	2	10.9357	17.1204	4.1376	0.9646
	5.2	10.4295	18.8033	4.3362	1.0600
	7	10.3977	18.9889	4.3576	1.0684
9	1	16.6625	31.8103	5.6400	1.0154
	2	15.1417	32.8225	5.7290	1.1350
	5.2	14.4409	36.0489	6.0040	1.2473
	7	14.3969	36.4048	6.0336	1.2572

6. *Estimation of the Parameters*

In this section, estimates of the two parameters (λ, α) of the Double weighted Rayleigh distribution are estimated and obtained. Method of moment (MOM) and maximum likelihood estimators (MLE) are presented.

6.1 Method of Moments

Let $x_1, x_2, \ldots, x_n$ be an independent random sample from the DWRD with parameters θ and *c*. The method of moment estimators are obtained by computing the population moments and equating to sample moments i,e

$$E(x^r)=\frac{1}{n}\sum_{i=1}^{n} x_i^{\,r}, r=1,2$$

The following equations are obtained using the first and second sample moments.

$$\frac{2^{\frac{3}{2}}\theta\left(\left(1+c^2\right)^2-1\right)}{\sqrt{\pi}\left(\left(1+c^2\right)^{\frac{3}{2}}-1\right)\left(1+c^2\right)^{\frac{1}{2}}}=\frac{1}{n}\sum_{i=1}^{n} x_i=\bar{x} \qquad (10)$$

$$\frac{3\theta^2\left(\left(1+c^2\right)^{\frac{5}{2}}-1\right)}{\left(\left(1+c^2\right)^{\frac{3}{2}}-1\right)\left(1+c^2\right)}=\frac{1}{n}\sum_{i=1}^{n} x_i^{\,2} \qquad (11)$$

Solving the two equations (10) and (11) simultaneously (numerical method), we will get $\hat{\theta}$ and $\hat{c}$ as estimate of θ and *c* respectively.

Note that from equation (10) when c is known we obtain estimate for θ which is given by

$$\hat{\theta}=\frac{\sqrt{\pi}\,\bar{x}\left(\left(1+c^2\right)^{\frac{3}{2}}-1\right)\left(1+c^2\right)^{\frac{1}{2}}}{2^{\frac{3}{2}}\,\theta\left(\left(1+c^2\right)^2-1\right)}$$

6.2 Maximum Likelihood Estimators

Let $x_1, x_2, ..., x_n$ be an independent random sample from the DWRD, then the likelihood function of DWRD is given by

$$L((x;\theta,c)=\prod_{i=1}^{n}\left[\sqrt{\frac{2}{\pi}}\frac{\left(1+c^2\right)^{\frac{3}{2}}}{\left(\left(1+c^2\right)^{\frac{3}{2}}-1\right)}\frac{x^2}{\theta^3}\exp\left(-\frac{x^2}{2\theta^2}\right)\left\{1-\exp\left(-\frac{(cx)^2}{2\theta^2}\right)\right\}\right]$$

and the log-likelihood function is given by

$$\log L(x;\theta,c)=\frac{n}{2}\log\left(\frac{2}{\pi}\right)+\frac{3n}{2}\log\left(1+c^2\right)-n\log\left(\left(1+c^2\right)^{\frac{3}{2}}-1\right)+2\sum_{i=1}^{n}\log x_i-3n\log\theta-\frac{\sum_{i=1}^{n}x_i^{\,2}}{2\theta^2}$$
$$+\sum_{i=1}^{n}\log\left\{1-\exp\left(-\frac{(cx_i)^2}{2\theta^2}\right)\right\}$$

By taking the derivatives of above equation with respect to the parameters θ and c, we obtain the following equations:

$$\frac{\partial\log L(x;\theta,c)}{\partial\theta}=-\frac{3n}{\theta}+\frac{\sum_{i=1}^{n}x_i^{\,2}}{\theta^3}-\sum_{i=1}^{n}\frac{\frac{(cx_i)^2}{\theta^3}\exp\left(-\frac{(cx_i)^2}{2\theta^2}\right)}{\left\{1-\exp\left(-\frac{(cx_i)^2}{2\theta^2}\right)\right\}}$$

$$\frac{\partial\log L(x;\theta,c)}{\partial c}=\frac{3nc}{\left(1+c^2\right)}-\frac{3nc\left(1+c^2\right)^{\frac{3}{2}}}{\left(\left(1+c^2\right)^{\frac{3}{2}}-1\right)}+\sum_{i=1}^{n}\frac{\frac{c\,x_i^{\,2}}{\theta^2}\exp\left(-\frac{(cx_i)^2}{2\theta^2}\right)}{\left\{1-\exp\left(-\frac{(cx_i)^2}{2\theta^2}\right)\right\}}$$

Equating these equations to zero, we get the normal equations

$$-\frac{3n}{\theta}+\frac{\sum_{i=1}^{n}x_i^{\,2}}{\theta^3}-\sum_{i=1}^{n}\frac{\frac{(cx_i)^2}{\theta^3}\exp\left(-\frac{(cx_i)^2}{2\theta^2}\right)}{\left\{1-\exp\left(-\frac{(cx_i)^2}{2\theta^2}\right)\right\}}=0 \qquad (12)$$

$$\frac{3nc}{\left(1+c^2\right)}-\frac{3nc\left(1+c^2\right)^{\frac{3}{2}}}{\left(\left(1+c^2\right)^{\frac{3}{2}}-1\right)}+\sum_{i=1}^{n}\frac{\frac{c\,x_i^{\,2}}{\theta^2}\exp\left(-\frac{(cx_i)^2}{2\theta^2}\right)}{\left\{1-\exp\left(-\frac{(cx_i)^2}{2\theta^2}\right)\right\}}=0 \qquad (13)$$

To find the maximum likelihood estimators of θ and c we have solve the above system of non-linear equations with respect to θ and c. Since this system has no closed form solution. Thus we use a iterative technique such as Newton-Raphson method to obtain the solution.

7. Numerical Example

This section illustrates the usefulness of the Double weighted Rayleigh distribution to real data to see how the new model works in practice. The real data set corresponds to the exceedances of flood peaks (in m^3 /s) of the Wheaton river near Carcross in Youkon territory, Canada. The data consist of 72 exceedances for the years 1952-1984, rounded to one decimal. The data was analyzed by Akinsete et al. [1] and is given below

1.7,2.2,14.4,1.1,0.4,20.6,5.3,0.7,1.9,13.0,12.0,9.3,1.4,18.7,8.5,25.5,11.6,14.1,22.1,1.1,2.5,14.4,1.7,37.6,0.6,2.2,39.0,0.3,15.0,11.0,7.3,22.9,1.7,0.1,1.1,0.6,9.0,1.7,7.0,20.1,0.4,2.8,14.1,9.9,10.4,10.7,30.0,3.6,5.6,30.8,13.3,4.2,25.5,3.4,11.9,21.5,27.6,36.4,2.7,64.0,1.5,2.5,27.4,1.0,27.1,20.2,16.8,5.3,9.7,27.5,2.5,27.0.

The data can be modeled by Double weighted Rayleigh distribution and also we estimate the parameters θ and c using Newton Raphson method beginning with the initial estimates $\hat{\theta} = 0.004$ and $\hat{c} = 0.03$. Then the estimate values are: $\hat{\theta} = 17.3807$ and $\hat{c} = 0.0299$, for 2 iterated.

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